reduced. By this choice of τ, the accuracy of the solution obtained can be maintained more uniformly over the whole of the calculated time interval and, as a result, great economy of the calculations can be achieved.

NOTATION

t, time; φ, turbulence function; ψ, current function; θ, dimensionless temperature; u, v, x, and y, components of velocity; L, scale of length; ΔT, characteristic temperature difference; ν, kinematic viscosity; $\varkappa$, thermal conductivity; g, acceleration due to gravity; β, thermal-expansion coefficient; τ, discretization step for time variable; h, discretization step for space variables; $\psi_{\bar{x}x} = (\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j})/h^2$; n, vector normal to surface; σ, iterational parameter; η, smoothing parameter for boundary condition on φ; s, number of internal iteration; s1 number of external iteration; ε_ψ, accuracy of internal iteration; ε_φ, accuracy of external iteration; $Pr = \nu/\varkappa$, Prandtl number; $Gr = g\beta L^3\Delta T/\nu^2$, Grashof number; Ra = PrGr, Rayleigh number; $R_{i\pm 1/2} = 0.5\rho h|u_{i\pm 1,j}|$, Reynolds difference number.

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HEAT-CONDUCTION PROBLEM FOR A MULTIPLY CONNECTED BODY

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A method for solving a heat-conduction problem for multiply connected domains is proposed based on consecutive solution of problems for doubly connected domains. To provide an example the heat-conduction problem is solved for a circle with two circular holes.

In applied mathematics the evaluation of temperature fields in multiply connected domains is a very difficult problem. As mentioned in [1] there is no universal analytic method which would ensure a solution to a heat-conduction problem. The possibilities of numerical methods are wide; their implementation, however, meets with difficulties, and to overcome them one must, as a rule, analyze each problem separately. An approach which would reduce the solution of a heat-conduction problem for a multiply connected domain to the solving of several problems of the same kind would, therefore, be welcome. The method proposed

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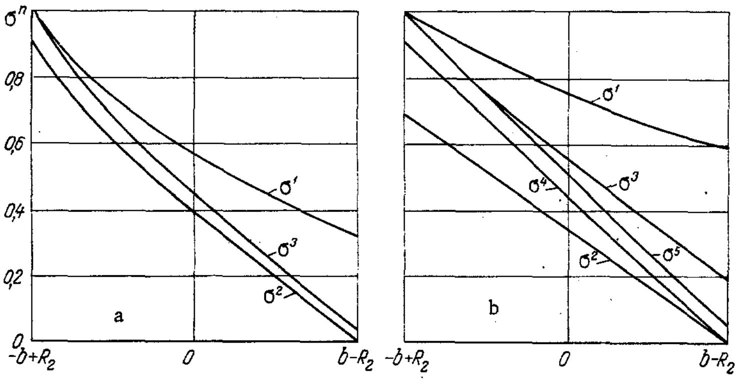

Fig. 1. Values of consecutive approximations σ^n on the straight line $b + R_2 \leq b - R_2$: a) for $R_1 = 1$, $R_2 = 0.25$, and $b = 0.5$; b) for $R_1 = 1$, $R_2 = 0.333$, and $b = 0.467$.

below is based on a consecutive solution of an appropriate problem for doubly connected domains. To make the exposition more clear the main concept of the method and the convergence proof are formulated for a triply connected domain only. In principle, there are no difficulties in extending this approach to the case of any multiply connected domains.

1. The first boundary-value problem is considered for a triply connected domain bounded by an outer closed curve Γ_0 and by inner ones Γ_1, Γ_2:

$$\Delta T(x, y) = f(x, y),\ T|_{\Gamma_0} = \varphi_0(s),\ T|_{\Gamma_i} = \varphi_i(s),\ i = 1,\ 2. \tag{1}$$

The required function T(x,y) is obtained with the aid of the following iterative process: Let the first approximation $U^1(x,y)$ be the solution for the doubly connected domain bounded by the outer closed curve Γ_0 and by an inner one, namely, Γ_1:

$$\Delta U^1(x, y) = f(x, y),\ U^1|_{\Gamma_0} = \varphi_0(s),\ U^1|_{\Gamma_1} = \varphi_1(s); \tag{2}$$

in other words, the connectivity of the domain due to the closed curve Γ_2 is ignored. Since the boundary condition is not satisfied on the last curve, it seems appropriate that for the second approximation $U^2(x,y)$ one adopts the solution of the problem

$$\Delta U^2(x, y) = 0,\ U^2|_{\Gamma_0} = 0,\ U^2|_{\Gamma_2} = \varphi_2(s) - U^1|_{\Gamma_2}. \tag{3}$$

The subsequent iterative procedure follows the pattern

$$\Delta U^k(x, y) = 0,\ U^k|_{\Gamma_0} = 0,\ U^k|_{\Gamma_i} = -U^{k-1}|_{\Gamma_i}, \tag{4}$$

where i = 1 for odd k, but i = 2 for even k.

It will be shown that the sum of the series

$$T(x, y) = \sum_{k=1}^{\infty} U^k(x, y) \tag{5}$$

is a solution of problem (1).

To this end it is assumed that the sequence $U^k(x, y)$ approaches zero over the entire domain for $k \to \infty$ and one considers the difference

$$S^N(x, y) = T(x, y) - \sum_{k=1}^{N} U^k(x, y).$$

The sequence of harmonic functions $S^N(x, y)$ vanishes at the outer closed curve and approaches zero uniformly at the inner ones. By the Harnack theorem [2] $S^N(x, y)$ approaches zero uniformly over the entire domain, that is, the relation (5) is satisfied.

It will now be shown that the sequence $U^k(x, y)$ approaches zero. For simplicity, one assumes that $\varphi_2(s) \equiv 0$ and $\varphi_1(s) \leq 0$. Then $U^{2k-1}(x, y) \leq 0$, $U^{2k}(x, y) \geq 0$. The following notation is now introduced:

$$a_1^{2k} = \max_{x,y\in\Gamma_1} U^{2k}(x, y),\ b_2^{2k-1} = \min_{x,y\in\Gamma_2} U^{2k-1}(x, y).$$

It follows from the maximum principle that the sequences a_1^{2k} and $|b_2^{2k-1}|$ are monotonically decreasing with $|b_2^{2k-1}| > a_1^{2k}$.

Since these sequences are bounded, there exists a limit

$$\lim_{k\to\infty} b_2^{2k-1} = -\lim_{k\to\infty} a_1^{2k} = B \leqslant 0.$$

It will now be shown that B = 0. Let us consider a sequence of auxiliary functions:

$$V^{2k-1}(x, y) = U^{2k-1}(x, y) - B + \varepsilon_1^{2k-1},$$

where $\varepsilon_1^{2k-1} = B - a_1^{2k-2}$ ($\varepsilon_1^{2k-1} \geq 0$ and $\varepsilon_1^{2k-1} \to 0$ for $k \to \infty$). It is not difficult to see that the functions $V^{2k-1}(x, y)$ are nonnegative and, consequently, they satisfy the Harnack inequality [2]. As the center of the circle C_ρ one adopts the point (x_{2k-1}, y_{2k-1}) of the minimum of $U^{2k-1}(x, y)$ on the "absent" boundary Γ_2

$$U^{2k-1}(x_{2k-1}, y_{2k-1}) = b_2^{2k-1}.$$

Hence one obtains

$$U^{2k-1}(x_{2k-1}, y_{2k-1}) = B - \varepsilon_2^{2k-1},$$

with $\varepsilon_2^{2k-1} \geq 0$, $\varepsilon_2^{2k-1} \to 0$ and $\varepsilon_1^{2k-1} \geq \varepsilon_2^{2k-1}$.

The radius R of the circle is selected in such a way that for any point of Γ_2 a circle of the radius R lies entirely in our domain. Assuming that $\rho < R$, one has for all points of the circle C_ρ

$$(\varepsilon_1^{2k-1} - \varepsilon_2^{2k-1}) \frac{R-\rho}{R+\rho} \leqslant V^{2k-1}(x, y) \leqslant \frac{R+\rho}{R-\rho} (\varepsilon_1^{2k-1} - \varepsilon_2^{2k-1}).$$

If as the center of the new circle, the arc end on Γ_2 is adopted which is cut off by C_ρ, and if one applies again the Harnack inequality, one obtains for the points of the new circle

$$0 \leqslant V^{2k-1}(x, y) \leqslant C\varepsilon, \quad C = \frac{R+\rho}{R-\rho}, \quad \varepsilon = (\varepsilon_1^{2k-1} - \varepsilon_2^{2k-1}) C.$$

By continuing this procedure one obtains for any point of Γ_2

$$0 \leqslant V^{2k-1}(x, y) \leqslant C^{\nu}\varepsilon,$$

where ν is the number of arcs into which the closed curve Γ_2 is subdivided by the circumference of radius ρ.

Thus, the sequence $V^{2k-1}(x, y)$ approaches uniformly zero on Γ_2 and, correspondingly, the sequence $U^{2k-1}(x, y)$ uniformly approaches B on Γ_2. It follows from (4) and the Harnack theorem that the sequence $U^{2k}(x, y)$ converges uniformly in the entire domain to a function $T^*(x, y)$ which is a solution of the problem

$$\Delta T^*(x, y) = 0, \quad T^*|_{\Gamma_0} = 0, \quad T^*|_{\Gamma_2} = -B.$$

It is shown in precisely the same way that the sequence of functions $U^{2k-1}(x, y)$ converges uniformly in the entire domain to a function $T^{**}(x, y)$ which is the solution of the problem

$$\Delta T^{**}(x, y) = 0, \quad T^{**}|_{\Gamma_0} = 0, \quad T^{**}|_{\Gamma_1} = B.$$

From the manner the sequences $U^k(x, y)$ have been obtained it now follows that if $B \neq 0$, then the strengthened maximum principle is invalid for the functions $T^*(x, y)$ and $T^{**}(x, y)$. Consequently, B = 0 and the relation (5) has been proved.

2. By way of example one obtains the solution of the following problem:

$$\Delta T(x, y) = 0, \quad T|_{\Gamma_0} = T|_{\Gamma_2} = 0, \quad T|_{\Gamma_1} = T_0, \tag{6}$$

where the boundaries Γ_0, Γ_1, Γ_2 are described by the equations

$$x^2 + y^2 = R_1^2, \quad (x+b)^2 + y^2 = R_2^2, \quad (x-b)^2 + y^2 = R_2^2.$$

The first approximation U^1 is given in accordance with the described iteration procedure by the solution of the problem

$$\Delta U^1(x, y) = 0, \quad U^1|_{\Gamma_0} = 0, \quad U^1|_{\Gamma_1} = T_0.$$

The variables are now changed by $x' = x + b_1$, $y' = y$, where

$$b_1 = \frac{1}{2}\left(\frac{R_1^2 - R_2^2}{b} + b\right), \quad b_2 = \frac{1}{2}\left(\frac{R_1^2 - R_2^2}{b} - b\right).$$

Now the conformal mapping

$$\xi = \frac{z+c}{z-c}, \quad z = x' + iy', \quad c = \sqrt{b_1^2 - R_1^2} = \sqrt{b_2^2 - R_2^2}$$

maps a not-concentric annulus into a concentric one with the radii $\gamma_1 = (b_1 + c)/R_1$, $\gamma_2 = [(b_2+c)/R_2]$ $(\gamma_2 > \gamma_1 > 1)$.

The function U^1 is given by [3]

$$U^1 = \beta \ln \frac{\rho}{\gamma_1}, \quad \beta = \frac{T_0}{\ln \frac{\gamma_2}{\gamma_1}}. \tag{7}$$

In the above, ρ, φ are polar coordinates on the ξ plane which can be expressed in terms of x and y as follows:

$$\rho = \sqrt{\frac{(x+b_1+c)^2 + y^2}{(x+b_1-c)^2 + y^2}}, \quad \varphi = \operatorname{arctg}\left[-\frac{2cy}{(x+b_1)^2 + y^2 - c^2}\right].$$

To be able to find the second approximation $U^2(x, y)$, one has to have the value of the function $U^1(x, y)$ on the closed curve Γ_2. Substituting in (7) the equation of the closed curve Γ_2 and changing to the coordinates $x'' = b_1 - x$, $y'' = y$, one obtains after some easy transformations

$$U^1|_{\Gamma_2} = \beta\left\{\frac{1}{2}\ln\frac{2\gamma_1 R_1 b_1 - (2\gamma_1 R_1 - \gamma_2 R_2)x''}{\frac{2R_1 b_1}{\gamma_1} - \left(\frac{2R_1}{\gamma_1} - \frac{R_2}{\gamma_2}\right)x''} - \ln \gamma_1\right\}. \tag{8}$$

The second approximation U^2 is the solution of the problem

$$\Delta U^2(x, y) = 0, \quad U^2|_{\Gamma_0} = 0, \quad U^2|_{\Gamma_2} = -U^1|_{\Gamma_2}. \tag{9}$$

The conformal mapping $\eta = (z' + c)/(z' - c)$ $(z' = x' + iy'')$ transforms the not-concentric circle into a concentric one of radius γ_1', and the circle Γ_2 into one of radius γ_2. In the polar coordinates of r and ψ in the η plane the expression (8) becomes

$$U^1|_{\Gamma_2} = A + \frac{\beta}{2}\ln\frac{1 - 2\alpha\cos\psi + \alpha^2}{1 - 2\varkappa\cos\psi + \varkappa^2},$$

where $\alpha = \gamma_1 R_1/\gamma_2 b_1 < 1$ and the quantities A and $\varkappa$ are given by the formula $A = T_0$, $\varkappa = \gamma_2 R_1/\gamma_1 b_1$ if $\gamma_2 R_1 < \gamma_1 b_1$ and $A = \beta \ln(b_1/R_1)$, but $\varkappa = \gamma_1 b_1/\gamma_2 R_1$, if $\gamma_2 R_1 > \gamma_1 b_1$.

Using the series expansion [4]

$$2\ln(1 - 2\alpha\cos\psi + \alpha^2) = -\sum_{k=1}^{\infty}\frac{\alpha^k}{k}\cos k\psi,$$

one can easily obtain the solution of the problem (9) by using the method of the separation of variables:

$$U^2(x, y) = -\frac{A}{\ln\frac{\gamma_2}{\gamma_1}}\ln\frac{r}{\gamma_1} + \beta\sum_{k=1}^{\infty}\frac{\gamma_2^k(\alpha^k - \varkappa^k)}{k}\cdot\frac{r^{2k} - \gamma_1^{2k}}{\gamma_2^{2k} - \gamma_1^{2k}}\,\frac{\cos k\psi}{r^k},$$

$$r = \sqrt{\frac{(b_1 - x + c)^2 + y^2}{(b_1 - x - c)^2 + y^2}}, \quad \psi = \operatorname{arctg}\left[-\frac{2cy}{(b_1 - x)^2 + y^2 - c^2}\right]. \tag{10}$$

As above, to obtain the third approximation $U^3(x, y)$ the expression for $U^2(x, y)$ is found on the closed curve Γ_1. To this end the coordinates r and ψ are expressed by means of the coordinates ρ, φ on the closed curve Γ_1:

$$r|_{\Gamma_1} = \sqrt{\frac{\gamma_2^2(1 - 2\alpha\cos\varphi + \alpha^2)}{1 - 2\varkappa\cos\varphi + \varkappa^2}}, \quad \psi|_{\Gamma_1} = \operatorname{arctg}\left[\frac{R_2 c^2 \sin\varphi}{2b_1(b_1 b_2 - c^2) - R_2(b_1^2 + R_1^2)\cos\varphi}\right].$$

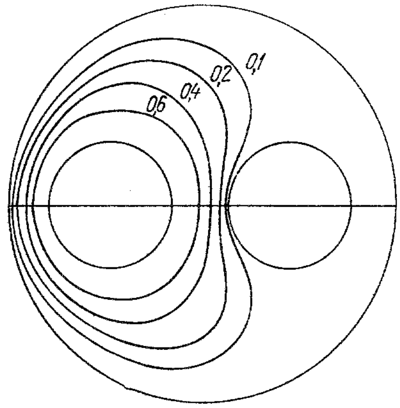

Fig. 2. Isotherm distribution for
$R_1 = 1$, $R_2 = 0.333$, and $b = 0.467$.

By substituting these expressions in (10) one finds the values of the function $U^2(x, y)$ on the closed curve Γ_1. For the third approximation $U^3(x, y)$ one considers the problem

$$\Delta U^3(x, y) = 0, \; U^3|_{\Gamma_0} = 0, \; U^3|_{\Gamma_1} = -U^2|_{\Gamma_1} = U^2(\varphi),$$

whose solution can be written as follows:

$$U^3(x, y) = a_0 \ln \frac{\rho}{\gamma_1} + \sum_{k=1}^{\infty} \frac{\gamma_2^k}{\rho^k} \cdot \frac{\rho^{2k} - \gamma_1^{2k}}{\gamma_2^{2k} - \gamma_1^{2k}} (a_k \cos k\varphi + b_k \sin k\varphi),$$

where a_k, b_k are the Fourier coefficients for the function $U^2(\varphi)$.

Further approximations are similarly obtained.

Computations with variously modified data were carried out to throw light on the convergence rate of the iteration procedure and other problems related to numerical implementation. As one would expect, the number of iterations needed to attain an acceptable accuracy depends essentially on how close the inner-closed curves are to each other. For example, for $R_1 = 1$, $R_2 = 0.25$, $b = 0.5$ one needs four approximations for the inequality $\max|U^N(x, y)| < 10^{-3}$ to hold but for $R_1 = 1$, $R_2 = 0.33$ and the eccentricity $b = 0.47$ one must already have six iterations for the same inequality to be satisfied. In Fig. 1a,b the distribution is shown for the first and second case of partial sums σ^n of the series (5) on the straight line which joins the centers of the circles Γ_1 and Γ_2 $(b + R_2 \leq x \leq b - R_2)$. In Fig. 2 the isotherms are shown for the second case.

In the general case a universal program was prepared in ALGOL for the electronic computer M-222 to solve heat-conduction problems for a domain of any connectivity and bounded by smooth closed curves. The "inner" problems (that is, intermediate problems for doubly connected domains) were solved by finite differences. The computations have shown that the proposed method is efficient only for domains of low connectivity containing not more than three to four inner closed curves. Any extension of connectivity results in rapid growth of the required computer time. The latter can be reduced if the exact solutions are known of the intermediate problems.

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